TRINITY COLLEGE

ADMISSIONS QUIZ (MATHEMATICS)

SPECIMEN TEST 3

There are ten questions below which are on various areas of mathematics. They are of varying levels of difficulty: some may be easy and others could be hard. You are not expected to answer all of them, or necessarily to give complete answers to questions. You should just attempt those that appeal to you, and they will be used as a basis for discussion in the interview that follows. You should bring what you have written with you to the interview.

1. By considering the graph of the function $f(x) = x^{-s}$ or otherwise, show that

$$\frac{1}{s-1} \leq 1 + 2^{-s} + 3^{-s} + \ldots \leq \frac{s}{s-1}$$

whenever $s > 1$.

2. Evaluate

$$\int_0^1 \frac{dx}{x + \sqrt{1 - x^2}} \ .$$

3. Sketch the graph of $\sqrt{y^2 - 1} = \frac{1}{2}(e^x - e^{-x})$ and also of $\sqrt{y^2 + 1} = \frac{1}{2}(e^x + e^{-x})$. In each case find an explicit formula for x in terms of y.

4. A fair coin is tossed repeatedly. Let t be the time at which we first see three consecutive heads (thus flips number $t - 2, t - 1, t$ are all heads), and let s be the time at which we first see four consecutive heads. What is the probability that $s = t + 1$? And what is the probability that $s = t + 9$?

5. 100 cards are on a table, with the numbers from 1 to 100 written on them. You are given a pile of 99 blank cards. You then repeat the following process. You remove two cards that are on the table, work out the difference between the two numbers written on those cards, write that down on a blank card, and put that card down on the table. At each step the total number of cards on the table goes down by 1, so eventually there will be just one card left. Prove that, however you carry out this process, the number written on that final card is even.

6. When written in decimal notation, every square number has at most 1000 digits that are not 0 or 1. True or false?

7. Find the smallest $a > 1$ such that

$$\frac{a + \sin x}{a + \sin y} \leq \exp(y - x)$$

for all $x \leq y$. [*Here* $\exp(x)$ *is another notation for* e^x.]

8. A total of 64 dots are arranged in an 8×8 grid. How many squares, with sides parallel to the axes, may be formed from these dots? And how many squares altogether?

9. Consider the differential equation

$$-\frac{d^2 y}{dx^2} + x^2 y = y.$$

(i) Verify that this equation is solved by $y = e^{-x^2/2}$.
(ii) Show by use of the chain rule that for any function $z(x)$

$$(\pm\frac{d}{dx} + x)(\mp\frac{d}{dx} + x)z = -\frac{d^2 z}{dx^2} + x^2 z \pm z.$$

  [*You will need to explain what the left hand side means. The choice of signs is intended to indicate that there are two identities for you to prove.*]
(iii) Hence or otherwise show that if y is a solution of the original differential equation then $w(x) = -\frac{dy}{dx} + xy$ satisfies the equation

$$-\frac{d^2 w}{dx^2} + x^2 w = 3w.$$

10. A cylindrical spaceship of mass M and cross-sectional area A is coasting at constant velocity when it suddenly encounters a dust cloud. The captain is dismayed to find that the dust sticks to the spaceship. If the density of dust is ρ, how far does the ship travel before its velocity is reduced by half?