TRINITY COLLEGE

ADMISSIONS QUIZ (MATHEMATICS)

SPECIMEN TEST 2

There are ten questions below which are on various areas of mathematics. They are of varying levels of difficulty: some may be easy and others could be hard. You are not expected to answer all of them, or necessarily to give complete answers to questions. You should just attempt those that appeal to you, and they will be used as a basis for discussion in the interview that follows. You should bring what you have written with you to the interview.

1. Which is greater as n gets large, $f(n) = 2^{2^{2^n}}$ or $g(n) = 100^{100^n}$?

2. Let $I_n = \int_0^1 \frac{x^n\,dx}{\sqrt{x^3+1}}$. Show that $(2n-1)I_n + 2(n-2)I_{n-3} = 2\sqrt{2}$ for all $n \geq 3$. Then compute I_8.

3. Show that if four distinct points of the curve $y = 2x^4 + 7x^3 + 3x - 5$ are collinear then their average x-coordinate is some constant k. Find k.

4. By sketching appropriate graphs, find all solutions to the equation $x - 1 = (e-1)\log x$. Hence sketch the graph of $f(x) = e^x - x^e$. (Here $\log x$ denotes the logarithm to base e – you may be more used to the notation $\ln x$.)

5. Six identical-looking coins are in a box, of which five are unbiased, while the sixth comes up heads with probability 3/4 and tails with probability 1/4. Three coins are chosen from the box at random and removed. One of those three is chosen at random and tossed three times, coming up heads every time. Given this information
   (a) What is the probability that the final coin selected was the biased coin?
   (b) What is the probability that the biased coin is amongst the three coins removed from the box?

6. Show that $\cos(n\theta) = f_n(\cos\theta)$ for polynomials $f_n(x)$ satisfying

$$f_{n+1}(x) = 2xf_n(x) - f_{n-1}(x).$$

Find all the roots of $f_2(x) + f_3(x) = 0$, and write them in the form $\cos(\phi)$ for suitable ϕ.

[Please turn over]

7. Let a and n be integers greater than 1. Suppose that $a^n - 1$ is prime. Show that $a = 2$ and n is prime. What can you say about primes of the form $2^n + 1$?

8. Consider a regular pentagon with vertices (in clockwise order) A, B, C, D, E. Let A' be the point of intersection of BD and CE, let B' be the point of intersection of CE and DA, and so on. If the triangle $AC'D'$ has area 1, what is the area of the pentagon $A'B'C'D'E'$?

9. A hand of thirteen playing cards is dealt from a standard pack of fifty-two. Write down expressions (in terms of binomial coefficients) for the probabilities of the following happening:
(a) the hand contains exactly one king;
(b) the hand contains at least two queens;
(c) the hand contains the same number of kings as queens.
[A binomial coefficient is a number of the form $\binom{n}{k} = {}^nC_k = \frac{n!}{k!(n-k)!}$.]

10. Consider a mass m at position $x(t)$ on a rough horizontal table attached to the origin by a spring with constant k (restoring force $-kx$) and with a dry friction force f

$$\begin{cases} f = F & \text{if } \dot{x} < 0 \\ -F \leq f \leq F & \text{if } \dot{x} = 0 \\ f = -F & \text{if } \dot{x} > 0. \end{cases}$$

What is the range of x where the mass can rest? Show that if the mass moves, the maximum excursion decreases by $2F/k$ per half cycle. Discuss the motion.